

TIGHT BOUNDS ON DISCRETE QUANTITATIVE HELLY NUMBERS

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ABSTRACT. Given a subset S of \mathbb{R}^n , let $c(S, k)$ be the smallest number t such that whenever finitely many convex sets have exactly k common points in S , there exist at most t of these sets that already have exactly k common points in S . For $S = \mathbb{Z}^n$, this number was introduced by Aliev et al. [2014] who gave an explicit bound showing that $c(\mathbb{Z}^n, k) = \mathcal{O}(k)$ holds for every fixed n . Recently, Chestnut et al. [2015] improved this to $c(\mathbb{Z}^n, k) = \mathcal{O}(k \cdot (\log \log k) \cdot (\log k)^{-1/3})$ and provided the lower bound $c(\mathbb{Z}^n, k) = \Omega(k^{(n-1)/(n+1)})$.

We provide a combinatorial description of $c(S, k)$ in terms of polytopes with vertices in S and use it to improve the previously known bounds as follows: We strengthen the bound of Aliev et al. [2014] by a constant factor and extend it to general discrete sets S . We close the gap for \mathbb{Z}^n by showing that $c(\mathbb{Z}^n, k) = \Theta(k^{(n-1)/(n+1)})$ holds for every fixed n . Finally, we determine the exact values of $c(\mathbb{Z}^n, k)$ for all $k \leq 4$.

1. INTRODUCTION

Let $n \in \mathbb{N}$ denote the dimension of the ambient space \mathbb{R}^n . Doignon [18] obtained the following analog of the classical theorem of Helly [25]: If convex sets C_1, \dots, C_m ($m \in \mathbb{N}$) have no point of \mathbb{Z}^n in common, then there exists a subset I of $\{1, \dots, m\}$ with at most 2^n elements such that the sets C_i with $i \in I$ already have no point of \mathbb{Z}^n in common. It is not possible to replace 2^n by a smaller number. This result was rediscovered independently by Bell [11], Scarf [30] and Hoffman [26]. In this paper, we continue the recent studies in [1, 13, 17] on quantitative versions of Doignon's theorem. Our main object of interest is the following number:

Definition 1 (Quantitative Helly number). *Let $S \subseteq \mathbb{R}^n$ and $k \in \mathbb{N}_0$. We define the quantitative Helly number $c(S, k)$ as the smallest number $t \in \mathbb{N}_0$ satisfying the following:*

If convex sets C_1, \dots, C_m ($m \in \mathbb{N}$) have exactly k points of S in common, then there exists a subset I of $\{1, \dots, m\}$ with at most t elements such that the sets C_i with $i \in I$ already have exactly k points of S in common.

In the degenerate case that no such number t exists, let $c(S, k) := \infty$, and if there is no convex set that contains exactly k points of S , let $c(S, k) := -\infty$.

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It turns out that restricting C_1, \dots, C_m in Definition 1 to closed halfspaces gives an equivalent definition of $c(S, k)$; see Lemma 8 in Section 2. The number

$$h(S) := c(S, 0)$$

is called the *Helly number* of (the space) S ; see [2, 4, 6]. Doignon's theorem gives the equality $c(\mathbb{Z}^n, 0) = h(\mathbb{Z}^n) = 2^n$. Since $c(S, k)$ and by this also $h(S)$ is invariant under non-singular affine transformations, Doignon's theorem can also be formulated in a coordinate-free form in terms of lattices and their rank. The values of $c(S, k)$ for $S = \mathbb{R}^n$ correspond to the classical theorems of Helly $c(\mathbb{R}^n, 0) = h(\mathbb{R}^n) = n + 1$ and Steinitz $c(\mathbb{R}^n, 1) = 2n$ (for the latter, see the explanation given in [23]); one obviously has $c(\mathbb{R}^n, k) = -\infty$ for every $k \geq 2$. The study of $c(S, k)$ for $S = \mathbb{Z}^n$ is motivated by applications to integer linear programming and has become a very active research topic; see [1, 2, 4, 6, 15, 26] for more information and [9, 15, 16] for related algorithmic research.

In view of applications, it is interesting to describe the asymptotic behavior of $c(\mathbb{Z}^n, k)$ and to understand how $c(\mathbb{Z}^n, k)$ can be computed in concrete situations. Already in Bell's work [11] one can find a generalization of the inequality $h(\mathbb{Z}^n) \leq 2^n$ which can be formulated as

$$c(\mathbb{Z}^n, k) \leq (k + 2)^n.$$

Aliev et al. [1] introduced $c(\mathbb{Z}^n, k)$ explicitly and improved Bell's bound to

$$(1) \quad c(\mathbb{Z}^n, k) \leq \lceil 2(k + 1)/3 \rceil (2^n - 2) + 2.$$

Thus, the growth of $c(\mathbb{Z}^n, k)$ is at most linear in k . Recently Chestnut et al. [13] showed that this number grows only sublinearly in k by proving

$$(2) \quad c(\mathbb{Z}^n, k) \leq C \cdot k(\log \log k)(\log k)^{-1/3} \cdot 2^n$$

whenever $\log k > 1$ and $n \in \mathbb{N}$, where $C > 0$ is an (unknown) absolute constant. As a complement to the upper bound, Chestnut et al. [13] established the lower bound

$$(3) \quad c(\mathbb{Z}^n, k) = \Omega(k^{\frac{n-1}{n+1}})$$

for every fixed $n \in \mathbb{N}$ and showed that this lower bound is asymptotically tight for $n = 2$.

Our contribution. We study $c(S, k)$ in the case of discrete S , paying special attention to $S = \mathbb{Z}^n$. We call a set $S \subseteq \mathbb{R}^n$ *discrete* if every bounded subset of S is finite. Our first main result provides an exact ‘polytopal description’ of $c(S, k)$, which we use as a tool in the proofs of all the other results. Let $\mathcal{P}(S)$ be the set of all polytopes whose vertices belong to S . In particular, $\mathcal{P}(\mathbb{Z}^n)$ is the well-known family of *integral polytopes*. For $k \in \mathbb{N}_0$ we introduce

$$g(S, k) := \max \{ |\text{vert}(P)| : P \in \mathcal{P}(S), |S \cap P \setminus \text{vert}(P)| = k \}.$$

Here, as usual $\text{vert}(P)$ denotes the set of all vertices of P . Note that in degenerate cases, $g(S, k)$ can be $-\infty$ or ∞ . It turns out that the sequence $g(S, 0), g(S, 1), \dots$ determines the sequence $c(S, 0), c(S, 1), \dots$ completely:

Theorem 2. *Let $S \subseteq \mathbb{R}^n$ be discrete and $k \in \mathbb{N}_0$. Then, one has*

$$(4) \quad c(S, k) = \max \{ g(S, \ell) + \ell - k : \ell \in \{0, \dots, k\}, g(S, \ell) + \ell - k \geq 0 \}$$

Furthermore, the condition $c(S, k) > -\infty$ is equivalent to $k \leq |S|$, and under this condition, $c(S, k)$ can be represented recursively as

$$(5) \quad c(S, 0) = g(S, 0) \quad \text{and} \quad c(S, k) = \max \{ c(S, k-1) - 1, g(S, k) \} \quad \text{for } 0 < k \leq |S|.$$

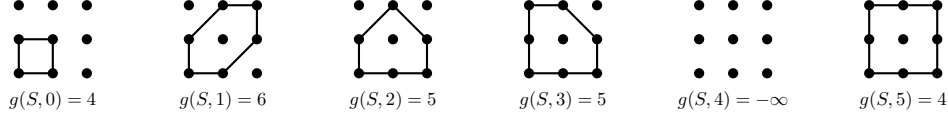


FIGURE 1. Values of $g(S, k)$ for $S = \{0, 1, 2\}^2$ and $k \in \{0, \dots, 5\}$ together with polytopes attaining these values. The respective values of $c(S, k)$ can be determined using (5); see also Figure 2.

The polytopal representation of the Helly number $h(S) = g(S, 0)$ provided in (5) was given in [15, Lem. 2.2]. Clearly, (5) implies $g(S, k) \geq g(S, k-1) - 1$ for all $k \in \mathbb{N}$ with $k \leq |S|$; this inequality was derived in [13] for $S = \mathbb{Z}^n$.

Theorem 2 shows that problems for $c(S, k)$ can be reworded as problems for polytopes in $\mathcal{P}(S)$. One can reformulate (4) without any use of $g(S, 0), \dots, g(S, k)$ as

$$(6) \quad c(S, k) = \max \{|S \cap P| - k : P \in \mathcal{P}(S), |S \cap P \setminus \text{vert}(P)| \leq k \leq |S \cap P|\}.$$

Representation (4) immediately implies the bounds

$$(7) \quad g(S, k) \leq c(S, k) \leq \max\{g(S, 0), \dots, g(S, k)\}.$$

Thus, one can bound $c(S, k)$ from above by bounding $g(S, 0), \dots, g(S, k)$. Moreover, if $g(S, k)$ turns out to be a largest value among $g(S, 0), \dots, g(S, k)$, one even has $c(S, k) = g(S, k)$. Using the upper bound in (7), we derive the following general upper bound on $c(S, k)$.

Theorem 3. *Let S be a discrete subset of \mathbb{R}^n with $|S| \geq 2$ and let $k \in \mathbb{N}_0$. Then one has*

$$c(S, k) \leq \lfloor (k+1)/2 \rfloor (h(S) - 2) + h(S).$$

Theorem 3 illustrates that the bound $c(S, k) = \mathcal{O}(k)$ follows from $h(S) < \infty$. Highlighting this message is one of the points of motivation for considering general discrete sets S . For $S = \mathbb{Z}^n$, Theorem 3 implies (1) and improves it by a constant factor for all $n \in \mathbb{N}$ and sufficiently large k .

The next two results are for $S = \mathbb{Z}^n$. The first one implies that the exact asymptotic order of $c(\mathbb{Z}^n, k)$ is $\Theta(k^{\frac{n-1}{n+1}})$ for every fixed $n \in \mathbb{N}$.

Theorem 4. *Let $n, k \in \mathbb{N}$. Then one has*

$$\left\lfloor (k/(2n))^{\frac{1}{n+1}} \right\rfloor^{n-1} \leq c(\mathbb{Z}^n, k) \leq (3n)^{5n} \cdot k^{\frac{n-1}{n+1}}.$$

The lower bound of Theorem 4 is a concrete version of (3), which we obtain using a short elementary argument. The upper bound can be considered as the main contribution of this paper.

Currently, the values of $c(\mathbb{Z}^n, k)$ are known exactly only in a few cases. Aliev et al. [1] observed that their inequality (1) holds with equality for $k = 0$ and $k = 1$, where the case $k = 0$ corresponds to Doignon's theorem. Chestnut et al. [13] showed that (1) still holds with equality for $k = 2$. Continuing this line of research, we determine $c(\mathbb{Z}^n, k)$ for $k = 3$ and $k = 4$. Thus, the five values of $c(\mathbb{Z}^n, k)$ which are now known exactly are as follows.

Theorem 5. *Let $n \geq 2$. Then one has*

$$c(\mathbb{Z}^n, 0) = 2^n, \quad c(\mathbb{Z}^n, 1) = c(\mathbb{Z}^n, 2) = c(\mathbb{Z}^n, 3) = 2^{n+1} - 2 \quad \text{and} \quad c(\mathbb{Z}^n, 4) = 2^{n+1}.$$

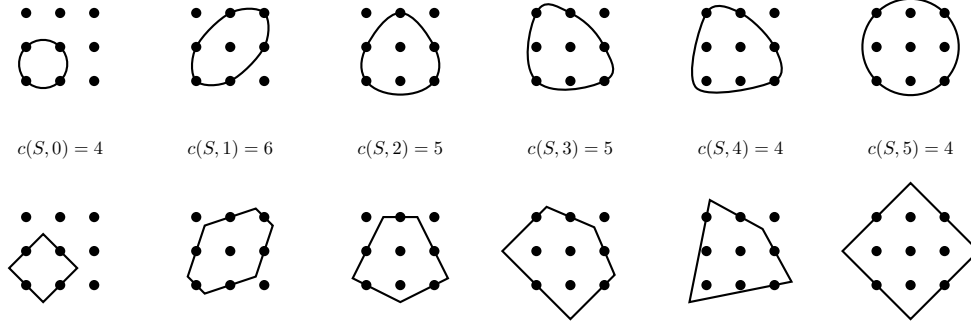


FIGURE 2. Values of $c(S, k)$ for $S = \{0, 1, 2\}^2$ and $k \in \{0, \dots, 5\}$ together with strictly convex bodies (see Corollary 6) and inclusion-maximal polytopes (see Theorem 7) attaining these values.

Applications. Starting from (6), one can use straightforward (but somewhat tedious) arguments for approximating polytopes by strictly convex sets in order to provide the following description of $c(S, k)$:

Corollary 6. *Let $S \subseteq \mathbb{R}^n$ be discrete and let $k \in \mathbb{N}_0$. Then $c(S, k)$ is the maximum number of points of S lying in the boundary of a strictly convex body that contains exactly k points of S in its interior.*

In view of this corollary, the equality $c(\mathbb{Z}^n, 1) = 2^{n+1} - 2$ from Theorem 5 implies an old result of Minkowski [21, Thm. 30.2] saying that every 0-symmetric strictly convex body with exactly one interior integer point contains at most $2^{n+1} - 1$ integer points in total. (Here the strict convexity is a crucial assumption; see [20] for an analogous investigation without this assumption.) Thus, Corollary 6 shows that the study of $c(\mathbb{Z}^n, k)$ can be viewed as a research in geometry of numbers dealing with the case of strictly convex bodies.

Yet another interpretation of $c(S, k)$ provides a link to the cutting plane theory for mixed-integer optimization problems:

Theorem 7. *Let $S \subseteq \mathbb{R}^n$ be discrete, let $k \in \mathbb{N}_0$ and $c(S, k) < \infty$. Then $c(S, k)$ is the maximum number of facets of an n -dimensional polyhedron that contains exactly k points of S in its interior and is inclusion-maximal with respect to this property.*

A discussion of the connections with the cutting plane theory and inclusion-maximal convex sets with k interior points in S is postponed to Appendix A.

Clearly, the number $\max\{c(S, 0), \dots, c(S, k)\}$ can be defined by replacing ‘exactly k points’ with ‘at most k points’ in Definition 1. This number has recently been introduced in [17, Def. 1.8] and coincides with $\mathbb{H}_S(k+1)$ in the notation of [17]. The inequalities in (7) imply that $\mathbb{H}_S(k+1)$ can be described as

$$\mathbb{H}_S(k+1) = \max\{c(S, 0), \dots, c(S, k)\} = \max\{g(S, 0), \dots, g(S, k)\}.$$

That is, $\mathbb{H}_S(k+1)$ is the maximum of $|\text{vert}(P)|$ taken over polytopes $P \in \mathcal{P}(S)$ satisfying $|S \cap P \setminus \text{vert}(P)| \leq k$. De Loera et al. [17] also introduced the so-called quantitative S -Tverberg number $\mathbb{T}_S(m, k)$ and provided the bound $\mathbb{T}_S(m, k) \leq \mathbb{H}_S(k)(m-1)kn + k$ on $\mathbb{T}_S(m, k)$ in terms of $\mathbb{H}_S(k)$. Combining the latter bound with our upper bounds on $c(S, k)$ we immediately obtain upper bounds on $\mathbb{T}_S(m, k)$ for general discrete sets S as well as $S = \mathbb{Z}^n$.

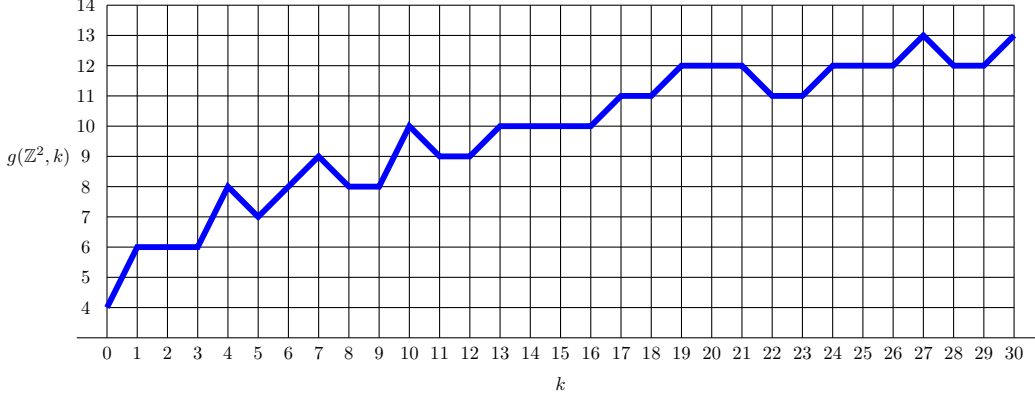


FIGURE 3. The values of $g(\mathbb{Z}^2, k) = c(\mathbb{Z}^2, k)$ for all $k \in \{0, \dots, 30\}$.

Open questions. We formulate several questions that arise naturally.

1. It is not clear whether sublinearity of $c(\mathbb{Z}^n, k)$ in k relies on specific properties of \mathbb{Z}^n or can be derived directly from the fact $h(\mathbb{Z}^n) < \infty$. This leads to the following question. If $S \subseteq \mathbb{R}^n$ is an arbitrary discrete set with $h(S) < \infty$, can the linear bound $c(S, k) = \mathcal{O}(k)$ be improved to a sublinear one?
2. One has $c(\mathbb{Z}^n, k) = \mathcal{O}(2^n)$ for every fixed $k \in \mathbb{N}_0$ and $c(\mathbb{Z}^n, k) = \mathcal{O}(k^{\frac{n-1}{n+1}})$ for every fixed $n \in \mathbb{N}$. What is the behavior of $c(\mathbb{Z}^n, k)$ when both k and n vary? Does one have $c(\mathbb{Z}^n, k) \leq C \cdot 2^n \cdot k^{\frac{n-1}{n+1}}$ for all $k, n \in \mathbb{N}$, where $C > 0$ is some absolute constant?
3. The example $S = \{0, 1, 2\}^2$ shows that $c(S, k)$ and $g(S, k)$ are not the same in general; see Figures 1 and 2. It is however not clear under which conditions on S one has the equality $c(S, k) = g(S, k)$ for every $k \in \mathbb{N}_0$. In particular, for $S = \mathbb{Z}^n$, the following question is open. Do there exist $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$ with $c(\mathbb{Z}^n, k) > g(\mathbb{Z}^n, k)$? Note that, in view of Theorem 5, one has $c(\mathbb{Z}^n, k) = g(\mathbb{Z}^n, k)$ for $n \geq 2$ and $k \leq 4$.

If for some $n \in \mathbb{N}$ there exists $k \in \mathbb{N}$ with $c(\mathbb{Z}^n, k) > g(\mathbb{Z}^n, k)$, then for the smallest such value $k = k'$ one has $c(\mathbb{Z}^n, k') > g(\mathbb{Z}^n, k')$ and $c(\mathbb{Z}^n, k' - 1) = g(\mathbb{Z}^n, k' - 1)$ so that (5) yields $g(\mathbb{Z}^n, k') < g(\mathbb{Z}^n, k' - 1) - 1$. This means that passing from $k = k' - 1$ to $k = k'$ the value $g(\mathbb{Z}^n, k)$ decreases by at least two. Thus, the question can be reformulated as follows: Do there exist $n \in \mathbb{N}$ and $k \in \mathbb{N}$ such that $g(\mathbb{Z}^n, k) \leq g(\mathbb{Z}^n, k - 1) - 2$?

Relying on Castryck's database [12, Rem. (3.4)] of all integral polygons with at most 30 interior integer points, we determine $g(\mathbb{Z}^2, k)$ for all $k \leq 30$ in a straightforward way; see Figure 3. Our computation verified that $g(\mathbb{Z}^2, k) \geq g(\mathbb{Z}^2, k - 1) - 1$ holds for every $k \in \{1, \dots, 30\}$. This implies that $g(\mathbb{Z}^2, k)$ and $c(\mathbb{Z}^2, k)$ coincide for $k \leq 30$. The computation also determined that, in the given range, $g(\mathbb{Z}^2, k)$ is not monotonic, since the equality $g(\mathbb{Z}^2, k) = g(\mathbb{Z}^2, k - 1) - 1$ is attained for $k \in \{5, 8, 11, 22, 28\}$ (for the case $k = 5$ see also [13]).

Structure, notation and terminology. In Sections 2–5 we present proofs of Theorems 2–5, respectively. Appendix A presents material related to Theorem 7, while in Appendix B

we determine an explicit constant in a result of Andrews [3] that is used in the proof of Theorem 4.

We use the notation $\mathbb{N}_0 := \{0, 1, 2, \dots\}$ and $\mathbb{N} := \{1, 2, 3, \dots\}$. Let $[m] := \{1, \dots, m\}$ for $m \in \mathbb{N}$ and $[0] := \emptyset$. For a set S , we denote by $|S|$ its cardinality.

We use basic background from the theory of polytopes, convex sets and geometry of numbers; see, for example, [8, 21, 22, 31]. Throughout, $n \in \mathbb{N}$ stands for the dimension of the ambient space \mathbb{R}^n , which is equipped with the standard inner product $\langle \cdot, \cdot \rangle$. For a polyhedron $P \subseteq \mathbb{R}^n$, we denote by $\text{vert}(P)$ and $\text{fct}(P)$ the set of vertices and facets of P , respectively, where a *facet* is a non-empty face of P of dimension $\dim(P) - 1$. Note that $\text{fct}(P) = \emptyset$ if P consists of one point only. For $S \subseteq \mathbb{R}^n$, we denote by $\mathcal{P}(S)$ the family of all polytopes P with $\text{vert}(P) \subseteq S$. We denote by $\text{int}(C)$, $\text{bd}(C)$, $\text{aff}(C)$, and $\text{conv}(C)$ the interior, the boundary, the affine hull, and the convex hull of $C \subseteq \mathbb{R}^n$, respectively. We say that $K \subseteq \mathbb{R}^n$ is a *convex body* if K is an n -dimensional compact convex set. By $\text{vol}(K)$ we denote the volume of a convex body K .

2. POLYTOPAL INTERPRETATION OF $c(S, k)$

In this section, we prove Theorem 2. In our argumentation, we will make use of the fact that one can define $c(S, k)$ equivalently by restricting C_1, \dots, C_m in Definition 1 to closed halfspaces.

Lemma 8. *Let S be a subset of \mathbb{R}^n (not necessarily a discrete one), let $k \in \mathbb{N}_0$ and $t \in \mathbb{N}_0$. Then the following two conditions are equivalent:*

- (i) *If convex sets C_1, \dots, C_m ($m \in \mathbb{N}$) have exactly k points of S in common, then there exists some $I \subseteq [m]$ with $|I| \leq t$ such that the sets C_i with $i \in I$ already have exactly k points of S in common.*
- (ii) *If closed halfspaces H_1, \dots, H_m ($m \in \mathbb{N}$) have exactly k points of S in common, then there exists some $I \subseteq [m]$ with $|I| \leq t$ such that the sets H_i with $i \in I$ already have exactly k points of S in common.*

Proof. In our proof we borrow ideas from [6]. As closed halfspaces are special convex sets, it is clear that (i) implies (ii). We verify the converse. Assume that (ii) is fulfilled. Consider convex sets C_1, \dots, C_m as in (i) such that the set

$$X := \bigcap_{i=1}^m C_i \cap S$$

has cardinality k . We first verify (i) in the case that C_1, \dots, C_m are polyhedra. In this case each C_i with $i \in [m]$ is the intersection of a finite family \mathcal{H}_i of closed halfspaces. By construction, $\bigcap_{H \in \mathcal{H}} H \cap S = X$ for $\mathcal{H} := \mathcal{H}_1 \cup \dots \cup \mathcal{H}_m$. Applying (ii) to the family \mathcal{H} we deduce the existence of $\mathcal{H}' \subseteq \mathcal{H}$ with $|\mathcal{H}'| \leq t$ such that $\bigcap_{H' \in \mathcal{H}'} H' \cap S = X$. Since $|\mathcal{H}'| \leq t$ and each $H' \in \mathcal{H}'$ contains some C_i with $i \in [m]$, there exists an index set $I \subseteq [m]$ with $|I| \leq t$ such that $\bigcap_{i \in I} C_i \cap S \subseteq \bigcap_{H' \in \mathcal{H}'} H' \cap S$. The left-hand side of the latter inclusion contains X , while the right-hand side coincides with X . Consequently, $\bigcap_{i \in I} C_i \cap S = X$.

It remains to verify (i) for general convex sets C_1, \dots, C_m . Let $I \subseteq [m]$ be an inclusion-minimal set with $\bigcap_{i \in I} C_i \cap S = X$. We need to verify $|I| \leq t$. The minimality of I implies that for every $i \in I$ there exists $s_i \in S$ that belongs to C_j with $j \in I \setminus \{i\}$ but does not belong to C_i . Consider the polytopes $C'_i := \text{conv}(X \cup \{s_j : j \in I \setminus \{i\}\})$ for $i \in I$. By construction, the set $\bigcap_{i \in I} C'_i \cap S$ coincides with X , while for every proper subset J of I , the set $\bigcap_{j \in J} C'_j \cap S$ is strictly larger than X . Since (i) has already been verified for polyhedra, it follows that $|I| \leq t$. \square

Proof of Theorem 2. For verifying (4), it suffices to verify (6), as both representations are equivalent. We denote by m the right-hand side of (6).

We first show $c(S, k) \geq m$. Consider an arbitrary polytope P as in (6) and let $t = |S \cap P| - k$. We need to show $c(S, k) \geq t$. One has $t \geq 0$ and the polytope P has at least t vertices. We choose t pairwise distinct vertices v_i with $i \in [t]$ of P and introduce the sets $C_i := \text{conv}(S \cap P \setminus \{v_i\})$. By construction, the set $X := \bigcap_{i=1}^t C_i \cap S = S \cap P \setminus \{v_1, \dots, v_t\}$ has cardinality t . On the other hand, for every proper subset I of $[t]$, the set $\bigcap_{i \in I} C_i \cap S$ strictly contains X as it additionally contains points v_i with $i \in [t] \setminus I$. This shows $c(S, k) \geq t$.

Next, we show $c(S, k) \leq m$. We distinguish the cases $c(S, k) = -\infty$, $c(S, k) = 0$, $c(S, k) \in \mathbb{N}$, and $c(S, k) = \infty$.

If $c(S, k) = -\infty$ there is nothing to prove. The case $c(S, k) = 0$ readily implies $|S| = k$ (note that the intersection of an empty family of convex subsets of \mathbb{R}^n is the whole space \mathbb{R}^n). In this case, The polytope $P = \text{conv}(S)$ satisfies $|S \cap P \setminus \text{vert}(P)| \leq k = |S \cap P|$. Therefore, $m \geq 0$.

Now, we consider the case $c(S, k) \in \mathbb{N}$ and let $t = c(S, k)$. By Lemma 8, there exist $a_1, \dots, a_t \in \mathbb{R}^n$ and $\beta_1, \dots, \beta_t \in \mathbb{R}$ such that the system

$$(8) \quad \langle a_1, x \rangle \leq \beta_1, \dots, \langle a_t, x \rangle \leq \beta_t$$

of t inequalities has exactly k solutions in S and such that, for every $i \in [t]$, there exists a point $s_i \in S$ that satisfies all but the i -th inequality of (8). Let X be the set of solutions of (8) that lie in S . We introduce the finite subset $S' := \text{conv}(X \cup \{s_1, \dots, s_t\}) \cap S$ of S and consider t parameters $\gamma_1, \dots, \gamma_t \in \mathbb{R}$. We first choose γ_i to be slightly larger than β_i for every $i \in [t]$. Since S' is finite, with this choice, the system

$$(9) \quad \langle a_1, x \rangle < \gamma_1, \dots, \langle a_t, x \rangle < \gamma_t$$

of t strict inequalities has the same set X of solutions in S' as the system (8). Furthermore, for each $i \in [t]$, the point $s_i \in S'$ satisfies all but the i -th inequality of (9). Now, for each $i \in [t]$, we can enlarge γ_i until we reach a value for which there exists a point $s'_i \in S' \setminus X$ that satisfies all but the i -th inequality of (9) and also satisfies the equality $\langle a_i, s'_i \rangle = \gamma_i$. Adjusting the values of $\gamma_1, \dots, \gamma_t$ consecutively using the above procedure, we ensure that for the points $s'_1, \dots, s'_t \in S'$ one has $\langle a_i, s'_j \rangle < \gamma_i$ and $\langle a_i, s'_i \rangle = \gamma_i$ for all $i, j \in [t]$ with $i \neq j$. Thus the points s'_1, \dots, s'_t are t distinct vertices of the polytope $P = \text{conv}(X \cup \{s'_1, \dots, s'_t\})$ and P contains exactly $|X| + t = k + t$ points of S in total. Thus, P occurs in the set on the right-hand side of (6) and for P one has $|S \cap P| - k = t$. Consequently, $c(S, k) = t \leq m$.

In the case $c(S, k) = \infty$, no choice of $t \in \mathbb{N}$ fulfills the property in Definition 1. Thus, we can apply the arguments of the previous case for every $t \in \mathbb{N}$. This yields $t \leq m$ for every $t \in \mathbb{N}$. Hence $m = \infty$.

Above, we have verified the validity (6) (and by this, also (4)).

We verify the equivalence of $c(S, k) > -\infty$ and $k \leq |S|$. One obviously has $c(S, k) = -\infty$ if $k > |S|$. If $k \leq |S|$ we can choose k distinct points $s_1, \dots, s_k \in S$. Then $S' = \text{conv}(\{s_1, \dots, s_k\}) \cap S$ is a finite set consisting of at least k points. Clearly, there exists a hyperplane separating exactly k of these points from the remaining ones. This yields the existence of a convex set with exactly k points in S and concludes the proof of the equivalence.

It remains to verify the recursive representation (5) under the condition $k \leq |S|$. The equality $c(S, 0) = g(S, 0)$ is a special case of (4). Let $0 < k \leq |S|$. One has $c(S, k) > -\infty$, which implies $c(S, k) \geq 0$. Thus, the maximum in (4) is non-negative so that one can omit the condition $g(S, \ell) + \ell - k \geq 0$ and represent $c(S, k)$ as the maximum of $g(S, \ell) + \ell - k$ taken

over $\ell \in \{0, \dots, k\}$. Analogously, $c(S, k-1)$ is also non-negative and can be given as the maximum of $g(S, \ell) + \ell - (k-1)$ taken over $\ell \in \{0, \dots, k-1\}$. From these representations of $c(S, k)$ and $c(S, k-1)$, (5) follows immediately. \square

3. AN UPPER BOUND FOR GENERAL DISCRETE SETS

In this section, we prove Theorem 3. To illustrate our approach for deriving the upper bound in Theorem 3, we first present a proof of the weaker bound

$$(10) \quad c(S, k) \leq (k+1)h(S)$$

for every $k \in \mathbb{N}$ and every non-empty discrete subset S of \mathbb{R}^n . Since the right-hand side of (10) is non-decreasing in k and $c(S, k)$ can be bounded in terms of $g(S, 0), \dots, g(S, k)$, it suffices to prove the bound $g(S, k) \leq (k+1)h(S)$. Consider an arbitrary $P \in \mathcal{P}(S)$ with $X = S \cap P \setminus \text{vert}(P)$ satisfying $|X| = k$. Let $X = \{s_1, \dots, s_k\}$. For a vector $u \in \mathbb{R}^n \setminus \{0\}$ and $i \in [k]$, we denote by H_i the hyperplane orthogonal to u and passing through s_i . If $u \in \mathbb{R}^n \setminus \{0\}$ is chosen generically, the hyperplanes H_1, \dots, H_k are pairwise distinct and one has $H_i \cap \text{vert}(P) = \emptyset$ for every $i \in [k]$. The hyperplanes H_1, \dots, H_k decompose \mathbb{R}^n into $k+1$ polyhedral regions: Two closed halfspaces and $k-1$ slabs (where a slab is the convex hull of two distinct parallel hyperplanes). We index these regions by $1, \dots, k+1$. For $i \in [k+1]$, by P_i we denote the convex hull of those points of $S \cap P$ that are contained in the i -th region. By construction $S \cap P_i = \text{vert}(P_i)$ and $\text{vert}(P) \subseteq \bigcup_{i=1}^{k+1} \text{vert}(P_i)$. Consequently,

$$|\text{vert}(P)| \leq \sum_{i=1}^{k+1} |\text{vert}(P_i)| \leq (k+1)h(S).$$

The rest of this section is devoted to proving Theorem 3 which improves the bound (10) by roughly a factor of $1/2$. In Lemma 9, given below, we estimate the number of vertices of a polytope $P \in \mathcal{P}(S)$ using the combinatorics of the polytope $Q := \text{conv}(S \cap P \setminus \text{vert}(P))$. We apply Lemma 9 in the cases $0 \leq \dim(Q) \leq 2$ for proving Theorem 3 in this section and Theorem 5 in Section 5.

Lemma 9. *Let $S \subseteq \mathbb{R}^n$ be discrete. Let $P \in \mathcal{P}(S)$ be such that $Q := \text{conv}(S \cap P \setminus \text{vert}(P))$ is non-empty and let $S' := \text{aff}(Q) \cap S$. Then*

$$|\text{vert}(P)| \leq 2(h(S) - h(S')) + |\text{fct}(Q)|(h(S') - \dim(Q)).$$

In particular, if $\dim(Q) \leq 1$, one has

$$(11) \quad |\text{vert}(P)| \leq 2h(S) - 2.$$

Proof. We distinguish several cases according to the dimension of Q . Consider the case $\dim(Q) = 0$, that is, Q consists of one point. For a generically chosen hyperplane H that contains Q one has $H \cap \text{vert}(P) = \emptyset$. The hyperplane H splits $\text{vert}(P)$ into two disjoint subsets V_1 and V_2 lying on different sides of H . The polytopes $P_\ell := \text{conv}(V_\ell \cup Q)$ with $\ell \in \{1, 2\}$ satisfy $S \cap P_\ell = \text{vert}(P_\ell) = V_\ell \cup Q$. Consequently,

$$|\text{vert}(P)| = |V_1| + |V_2| = |\text{vert}(P_1)| + |\text{vert}(P_2)| - 2 \leq 2h(S) - 2.$$

We switch to the case $1 \leq \dim(Q) \leq n-1$. Let $t := |\text{vert}(Q)|$, $s := |\text{fct}(Q)|$ and $V' := \text{vert}(P) \cap \text{aff}(Q)$. Let F_1, \dots, F_s be all facets of Q . Consider a point a in the relative interior of Q . The affine space $\text{aff}(Q)$ can be subdivided into s polyhedral cones, where for each $i \in [s]$, the i -th cone is defined as the union of all rays emanating from a and passing through points of F_i . Choosing the point a in the relative interior of Q generically, we ensure that none of the above cones contains points of V' in their relative boundary.

In the i -th cone we pick a subset X_i of $F_i \cap S$ of cardinality $\dim(Q)$ satisfying $\text{conv}(X_i) \cap S = X_i$. Let V'_i be the set of all points of V' contained in the i -th cone and let

$$P'_i := \text{conv}(V'_i \cup X_i).$$

By construction, P'_i satisfies $P'_i \cap S' = \text{vert}(P'_i) = V'_i \cup X_i$. We also introduce

$$m := \max \{ |\text{vert}(P'_i)| : i \in [s] \}.$$

Without loss of generality, let $m = |\text{vert}(P'_1)|$.

Consider a hyperplane H that contains $\text{aff}(Q)$. Choosing such a hyperplane H generically, we ensure $H \cap \text{vert}(P) = V'$. The hyperplane H splits $\text{vert}(P) \setminus V'$ into two disjoint subsets, say V_1 and V_2 . For each $\ell \in \{1, 2\}$, the polytope

$$P_\ell := \text{conv}(V_\ell \cup \text{vert}(P'_1))$$

has the property $S \cap P_\ell = \text{vert}(P_\ell) = V_\ell \cup \text{vert}(P'_1)$. Taking into account $|\text{vert}(P_\ell)| \leq h(S)$, for $\ell \in \{1, 2\}$, and $m \leq h(S')$, and $s \geq 2$, we obtain the desired bound:

$$\begin{aligned} |\text{vert}(P)| &= |V_1| + |V_2| + \sum_{i=1}^s |V'_i| \\ &\leq (|\text{vert}(P_1)| - m) + (|\text{vert}(P_2)| - m) + s(m - \dim(Q)) \\ &= |\text{vert}(P_1)| + |\text{vert}(P_2)| + (s - 2)m - s \dim(Q) \\ &\leq 2h(S) + (s - 2)h(S') - s \dim(Q) \\ &= 2(h(S) - h(S')) + |\text{fct}(Q)| (h(S') - \dim(Q)). \end{aligned}$$

The case $\dim(Q) = n$ is analogous to the previous one with the exception that no separating hyperplane H needs to be introduced. In the notation from the previous case, we have $V' = \text{vert}(P)$ and obtain

$$|\text{vert}(P)| = \sum_{i=1}^s |V'_i| \leq s(m - n) \leq |\text{fct}(Q)| (h(S) - n). \quad \square$$

Proof of Theorem 3. In view of (7), it suffices to verify the asserted inequality for $g(S, k)$ in place of $c(S, k)$. We assume $n \geq 2$ as for $n = 1$, one has $g(S, k) = h(S) = 2$ and the assertion is trivial. We proceed by induction on k . The case $k = 0$ corresponds to the inequality $g(S, 0) \leq h(S)$, which holds (with equality) in view of Theorem 2. It follows directly from Lemma 9, that $g(S, k) \leq 2h(S) - 2$, for $k \in \{1, 2\}$. This inequality is precisely what we need to prove in these cases.

Next, let $k \geq 3$ and assume that $g(S, k') \leq \lfloor (k' + 1)/2 \rfloor (h(S) - 2) + h(S)$ has been verified for every $k' \in \{0, \dots, k - 1\}$. Consider an arbitrary polytope P with $\text{vert}(P) \subseteq S$ and such that the set $X := S \cap P \setminus \text{vert}(P)$ consists of exactly k points. Since $k \geq 3$, the polytope $\text{conv}(X)$ has dimension at least one.

We pick an edge E of $\text{conv}(X)$ (where $E = \text{conv}(X)$ if $\text{conv}(X)$ is one-dimensional) and two consecutive points s_1 and s_2 of S lying in E . Since E is one-dimensional, the set

$$V' := \text{aff}(E) \cap \text{vert}(P)$$

consists of at most two points. A generically chosen hyperplane H with $H \cap \text{conv}(X) = E$ satisfies $H \cap \text{vert}(P) = V'$. The hyperplane H splits $\text{vert}(P) \setminus V'$ into two sets, say V_1 and V_2 , where we assume that V_2 lies on the same side of H as $\text{conv}(X)$. By construction,

$$P_1 := \text{conv}(V_1 \cup \{s_1, s_2\})$$

is a polytope with $S \cap P_1 = \text{vert}(P_1) = V_1 \cup \{s_1, s_2\}$. Furthermore,

$$P_2 := \text{conv}(V_2 \cup \{s_1, s_2\})$$

is a polytope with $V_2 \cup \{s_1, s_2\} = \text{vert}(P_2)$. One has $S \cap P_2 \setminus \text{vert}(P_2) \subseteq X \setminus \{s_1, s_2\}$ and so $k' := |S \cap P_2 \setminus \text{vert}(P_2)| \leq k - 2$.

Summarizing, we obtain the desired bound on $|\text{vert}(P)|$ using the induction assumption:

$$\begin{aligned} |\text{vert}(P)| &= |V_1| + |V_2| + |V'| \\ &\leq (|\text{vert}(P_1)| - 2) + (|\text{vert}(P_2)| - 2) + 2 \\ &= |\text{vert}(P_1)| + |\text{vert}(P_2)| - 2 \\ &\leq h(S) + \lfloor (k' + 1)/2 \rfloor (h(S) - 2) + h(S) - 2 \\ &= (\lfloor (k' + 1)/2 \rfloor + 1) (h(S) - 2) + h(S) \\ &\leq \lfloor (k + 1)/2 \rfloor (h(S) - 2) + h(S). \end{aligned} \quad \square$$

4. BOUNDS FOR THE INTEGER LATTICE

In this section, we prove Theorem 4. We first give a proof for the upper bound. To this end, let us recall three results in geometry of numbers. The first one relates the number of vertices of an integral polytope to its volume.

Theorem 10 (Andrews [3]). *For every $n \in \mathbb{N}$ there exists a constant $\alpha(n)$ such that for every n -dimensional polytope $P \in \mathcal{P}(\mathbb{Z}^n)$ one has*

$$|\text{vert}(P)| \leq \alpha(n) \cdot \text{vol}(P)^{\frac{n-1}{n+1}}.$$

The constant can be chosen as $\alpha(n) = (3n)^{4n}$.

We also refer to [7] for a discussion of various proofs of Theorem 10 available in the literature. A discussion of the choice of $\alpha(n)$ is given in Appendix B. Second, we need an upper bound on the volume of a convex body in terms of the so-called coefficient of asymmetry. Given a convex body $K \subseteq \mathbb{R}^n$ and $x \in \text{int}(K)$, the *coefficient of asymmetry* of K with respect to x can be defined as

$$\text{ac}(K, x) := \min \{ \lambda \geq 0 : x - K \subseteq \lambda(K - x) \},$$

see [24]. Using the cancelation law for Minkowski addition of convex bodies (see [31, p. 48]), the latter can be reformulated as

$$\text{ac}(K, x) = \min \left\{ \lambda \geq 0 : x + \frac{1}{\lambda + 1}(K - K) \subseteq K \right\}.$$

Lagarias and Ziegler [27] observe that the following result can be derived from an inequality of van der Corput; see also [22, p. 47 & p. 127].

Theorem 11 (Lagarias and Ziegler [27]). *Let $K \subseteq \mathbb{R}^n$ be a convex body and let $x \in \mathbb{Z}^n \cap \text{int}(K)$. Then, one has*

$$\text{vol}(K) \leq (1 + \text{ac}(K, x))^n \cdot |\mathbb{Z}^n \cap \text{int}(K)|.$$

Finally, we will make use of the following version of flatness theorems:

Theorem 12 ([8, Thm. 8.3]). *For every $n \in \mathbb{N}$ there exists a constant $\phi(n)$ such that for every convex body $K \subseteq \mathbb{R}^n$ with $K \cap \mathbb{Z}^n = \emptyset$ there exists a vector $u \in \mathbb{Z}^n \setminus \{0\}$ with*

$$w(K, u) := \max_{x \in K} \langle x, u \rangle - \min_{x \in K} \langle x, u \rangle \leq \phi(n).$$

The constant can be chosen as $\phi(n) = n^{5/2}$.

While flatness theorems providing better choices of $\phi(n)$ in the sense of asymptotic behavior are available in the literature, we prefer the bound $n^{5/2}$, because this bound is explicit and our subsequent estimates are not very sensitive with respect to the exponent $5/2$. The exponent $5/2$ can be replaced by a smaller one at the cost of introducing an unknown multiplicative constant.

Proof of $c(\mathbb{Z}^n, k) \leq (3n)^{5n} \cdot k^{\frac{n-1}{n+1}}$ in Theorem 4. Since the right-hand side of the asserted inequality is non-decreasing in k and $c(\mathbb{Z}^n, k)$ can be bounded by $g(\mathbb{Z}^n, 0), \dots, g(\mathbb{Z}^n, k)$, it suffices to prove the inequality with $g(\mathbb{Z}^n, k)$ in place of $c(\mathbb{Z}^n, k)$. Let $\beta(n) := (3n)^{5n}$. We proceed by induction on n . The inequality is trivial for $n = 1$. Let $n \geq 2$ and assume that the inequality has been verified in dimensions $1, \dots, n-1$. Let $P \subseteq \mathbb{R}^n$ be a polytope with $\text{vert}(P) \subseteq \mathbb{Z}^n$ and $|\mathbb{Z}^n \cap P \setminus \text{vert}(P)| = k$. We have to show

$$(12) \quad |\text{vert}(P)| \leq \beta(n) \cdot k^{\frac{n-1}{n+1}}.$$

As $\beta(n) \cdot k^{\frac{n-1}{n+1}}$ is non-decreasing in n , in the case $\dim(P) < n$, (12) follows from the induction assumption. Thus, we can assume $\dim(P) = n$.

It is known and not hard to prove that there exists a point $x \in \text{int}(P)$ with $\text{ac}(P, x) \leq n$; see [24]. For example, one can take x to be the center of the maximum volume simplex contained in K . We first consider the case that P contains a “deep” integer point, that is, there exists a point $y \in (\frac{1}{2}x + \frac{1}{2}P) \cap \mathbb{Z}^n$. Note that y is contained in the interior of P and that it satisfies $\text{ac}(P, y) \leq 2n+1$ since

$$\begin{aligned} y + \frac{1}{2(n+1)}(P - P) &\subseteq \frac{1}{2}x + \frac{1}{2}P + \frac{1}{2(n+1)}(P - P) \\ &= \frac{1}{2}P + \frac{1}{2}\left(x + \frac{1}{n+1}(P - P)\right) \\ &\subseteq \frac{1}{2}P + \frac{1}{2}P = P. \end{aligned}$$

Thus, by Theorem 11, we obtain

$$\text{vol}(P) \leq (2n+2)^n \cdot |\mathbb{Z}^n \cap \text{int}(P)| \leq (2n+2)^n \cdot k \leq (3n)^n \cdot k,$$

and (12) follows in view of Theorem 10.

We are left with the case that $\frac{1}{2}x + \frac{1}{2}P$ contains no integer point. Using Theorem 12 (and its notation), we derive that for some vector $u \in \mathbb{Z}^n \setminus \{0\}$ one has

$$w(P, u) = 2 \cdot w(\frac{1}{2}x + \frac{1}{2}P, u) \leq 2 \cdot \phi(n).$$

Thus, the set

$$I := \left\{ i \in \mathbb{Z} : \min_{p \in P} \langle p, u \rangle \leq i \leq \max_{p \in P} \langle p, u \rangle \right\}$$

satisfies $|I| \leq 2 \cdot \phi(n) + 1$.

For each $i \in I$ define $P_i := \text{conv}(\{p \in P \cap \mathbb{Z}^n : \langle p, u \rangle = i\})$ and $k_i := |\mathbb{Z}^n \cap P_i \setminus \text{vert}(P_i)|$. Note that

$$|\text{vert}(P)| \leq \sum_{i \in I} |\text{vert}(P_i)|.$$

Using the induction assumption, we get

$$|\text{vert}(P_i)| \leq g(n-1, k_i) \leq \beta(n-1) \cdot k_i^{\frac{n-2}{n}} \leq \beta(n-1) k^{\frac{n-2}{n}}$$

for each $i \in I$ with $k_i \geq 1$, while $|\text{vert}(P_i)| \leq 2^{n-1}$ holds for all $i \in I$ with $k_i = 0$. This yields $|\text{vert}(P_i)| \leq (\beta(n-1) + 2^{n-1}) k^{\frac{n-2}{n}}$ for every $i \in I$. Consequently,

$$|\text{vert}(P)| \leq |I|(\beta(n-1) + 2^{n-1}) k^{\frac{n-2}{n}} \leq (2 \cdot \phi(n) + 1)(\beta(n-1) + 2^{n-1}) k^{\frac{n-2}{n}}.$$

The latter implies (12) since one has $\frac{n-2}{n} \leq \frac{n-1}{n+1}$ and

$$(2 \cdot \phi(n) + 1) \cdot (\beta(n-1) + 2^{n-1}) \leq 3n^{5/2} \cdot 2\beta(n-1) \leq 6n^3 \cdot (3n)^{5n-5} \leq (3n)^{5n} = \beta(n). \quad \square$$

We now focus on the lower bound in Theorem 4.

Proof of $\lfloor (k/(2n))^{\frac{1}{n+1}} \rfloor^{n-1} \leq c(\mathbb{Z}^n, k)$ in Theorem 4. Consider two parameters $t, s \in \mathbb{N}$, which will be fixed later. We compute the number of vertices and the number of integer non-vertices of the integral polytope $P := \text{conv}(C \cap \mathbb{Z}^n)$, where C is a compact convex set given by

$$C := \{(x_1, \dots, x_n) \in [1, t]^{n-1} \times \mathbb{R} : \ell(x_1, \dots, x_{n-1}) \leq x_n \leq u(x_1, \dots, x_{n-1})\}$$

with

$$\begin{aligned} \ell(x_1, \dots, x_{n-1}) &:= \sum_{i=1}^{n-1} (x_i^2 - t^2), \\ u(x_1, \dots, x_{n-1}) &:= s + \sum_{i=1}^{n-1} (t^2 - x_i^2). \end{aligned}$$

Note that on $[1, t]^{n-1}$ the function ℓ is strictly smaller than u , the function ℓ is strictly convex and the function u is strictly concave. In view of this, a point (x_1, \dots, x_n) is a vertex of P if and only if x_1, \dots, x_{n-1} all belong to $\{1, \dots, t\}$ and x_n is either equal to $\ell(x_1, \dots, x_{n-1})$ or equal to $u(x_1, \dots, x_{n-1})$. Thus, we obtain

$$|\text{vert}(P)| = 2t^{n-1}.$$

Furthermore, a point (x_1, \dots, x_n) belongs to $P \cap \mathbb{Z}^n$ if and only if x_1, \dots, x_{n-1} all belong to $\{1, \dots, t\}$ and x_n is an integer value in $\{\ell(x_1, \dots, x_{n-1}), \dots, u(x_1, \dots, x_{n-1})\}$. Thus, we see that

$$|P \cap \mathbb{Z}^n| = \sum_{x_1, \dots, x_{n-1} \in [t]} (u(x_1, \dots, x_{n-1}) - \ell(x_1, \dots, x_{n-1}) + 1),$$

where

$$u(x_1, \dots, x_{n-1}) - \ell(x_1, \dots, x_{n-1}) + 1 = s + 1 + 2(n-1)t^2 - 2 \sum_{i=1}^{n-1} x_i^2.$$

For the determination of $|\mathbb{Z}^n \cap P \setminus \text{vert}(P)|$ we first note that

$$\begin{aligned} \sum_{x_1, \dots, x_{n-1} \in [t]} \sum_{i=1}^{n-1} x_i^2 &= (n-1) \sum_{x_1, \dots, x_{n-1} \in [t]} x_1^2 \\ &= (n-1)t^{n-2} \sum_{x_1 \in [t]} x_1^2 \\ &= (n-1)t^{n-2} \cdot \frac{1}{6}t(t+1)(2t+1) \\ &= \frac{1}{6}(n-1)(t+1)(2t+1)t^{n-1}. \end{aligned}$$

Hence, we have

$$\begin{aligned} |\mathbb{Z}^n \cap P \setminus \text{vert}(P)| &= |P \cap \mathbb{Z}^n| - |\text{vert}(P)| \\ &= (s + 2(n-1)t^2 - \frac{1}{3}(n-1)(t+1)(2t+1) - 1)t^{n-1} \\ &= \left(s + \frac{1}{3}(n-1)(4t^2 - 3t - 1) - 1 \right) t^{n-1}. \end{aligned}$$

In particular, for $s = 1$ one has $|\mathbb{Z}^n \cap P \setminus \text{vert}(P)| \leq 2nt^{n+1}$.

Next, we fix t and s . We choose t to be the largest integer with $2nt^{n+1} \leq k$, that is,

$$t := \left\lfloor \left(\frac{k}{2n} \right)^{\frac{1}{n+1}} \right\rfloor.$$

For the rest of the proof we assume that $k \geq 2n$, since otherwise the asserted inequality is trivial. By the choice of t , for $s = 1$ we have $|\mathbb{Z}^n \cap P \setminus \text{vert}(P)| \leq k$. Let s be the largest possible integer, for which the inequality $|\mathbb{Z}^n \cap P \setminus \text{vert}(P)| \leq k$ is fulfilled. Consider the cardinality

$$k' := \left(s + \frac{1}{3}(n-1)(4t^2 - 3t - 1) - 1 \right) t^{n-1}$$

of $|\mathbb{Z}^n \cap P \setminus \text{vert}(P)|$. Clearly, we have $k' \leq k$ by construction. Furthermore, $k - k' \leq t^{n-1}$. In fact, if we had $k - k' > t^{n-1}$, the parameter s could be enlarged by 1, contradicting the choice of s . Thus, by construction we have $k - k' \leq t^{n-1}$ and $g(\mathbb{Z}^n, k') \geq 2t^{n-1}$, and hence, using Theorem 2, we get

$$c(\mathbb{Z}^n, k) \geq g(\mathbb{Z}^n, k') + k' - k \geq 2t^{n-1} - t^{n-1} = t^{n-1}. \quad \square$$

5. SPECIFIC VALUES FOR THE INTEGER LATTICE

In this section, we prove Theorem 5. We determine the values $c(\mathbb{Z}^n, k)$ for $k \in \{1, \dots, 4\}$ by computing $g(\mathbb{Z}^n, k)$ for all k in this range. We start by providing lower bounds on $g(\mathbb{Z}^n, k)$:

Lemma 13. *Let $n, k \in \mathbb{N}$. Then $g(\mathbb{Z}^n, k) \geq 2^{n+1} - 2$. Furthermore, if $n \geq 2$, one has $g(\mathbb{Z}^n, 4) \geq 2^{n+1}$.*

Proof. It is straightforward to check that the integral polytope

$$P_n := \text{conv}([-1, 0]^n \cup [0, 1]^n)$$

satisfies $|\text{vert}(P_n)| = 2^{n+1} - 2$ and $\mathbb{Z}^n \cap P_n \setminus \text{vert}(P_n) = \{0\}$. This verifies $g(\mathbb{Z}^n, k) \geq 2^{n+1} - 2$ for $k = 1$ and every $n \in \mathbb{N}$. Consider $k \geq 2$. If $n = 1$, the assertion is trivial. So, let $n \geq 2$. The polytope P_{n-1} generates the prism $P_{n-1} \times [0, 1]$ which has exactly two non-vertex integer points, namely 0 and $u := (0, \dots, 0, 1)$. Adding the integer point $-u$ below 0 and the integer points $2 \cdot u, \dots, k \cdot u$ above u we generate the integral polytope

$$P = \text{conv}((P_{n-1} \times [0, 1]) \cup (\{0\}^{n-1} \times [-1, k]))$$

with $|\text{vert}(P)| = 2(2^n - 2) + 2 = 2^{n+1} - 2$ and k non-vertex integer points $0 \cdot u, \dots, (k-1) \cdot u$. This concludes the proof of $g(\mathbb{Z}^n, k) \geq 2^{n+1} - 2$ for any $k \in \mathbb{N}$.

For bounding $g(\mathbb{Z}^n, 4)$ we can now rely on the existence of a polytope $P' \in \mathcal{P}(\mathbb{Z}^{n-1})$ with $|\text{vert}(P')| = 2^n - 2$ and with $X' := \mathbb{Z}^{n-1} \cap P' \setminus \text{vert}(P')$ satisfying $|X'| = 2$ (just use the above considerations for $k = 2$). Similarly to the previous construction, we build the prism

$P' \times [0, 1]$ and add the points of $X' \times \{-1\}$ below $X' \times \{0\}$ and the points of $X' \times \{2\}$ above $X' \times \{1\}$. As a result, we obtain the polytope

$$P = \text{conv}((P' \times [0, 1]) \cup (X' \times [-1, 2])) \in \mathcal{P}(\mathbb{Z}^n)$$

satisfying $|\text{vert}(P)| = 2(2^n - 2) + 4 = 2^{n+1}$ and $|\mathbb{Z}^n \cap P \setminus \text{vert}(P)| = 4$. \square

Proof of Theorem 5. The equality $c(\mathbb{Z}^n, 0) = 2^n$ is precisely Doignon's theorem [18]. Since $g(\mathbb{Z}^n, k)$ is non-decreasing for $k \in \{0, \dots, 4\}$, we obtain, in view of (7), that $c(\mathbb{Z}^n, k) = g(\mathbb{Z}^n, k)$ holds for every $k \in \{1, \dots, 4\}$. Thus it suffices to determine $g(\mathbb{Z}^n, k)$ for $k \in \{1, \dots, 4\}$.

The desired lower bounds on $g(\mathbb{Z}^n, k)$ are provided by Lemma 13. We need to verify the matching upper bounds. To this end, consider an arbitrary polytope $P \in \mathcal{P}(\mathbb{Z}^n)$ such that the set $X := \mathbb{Z}^n \cap P \setminus \text{vert}(P)$ satisfies $|X| = k$. We introduce the polytope $Q := \text{conv}(X)$.

Let t be the number of vertices of Q . Since $1 \leq k \leq 4$, one has $1 \leq t \leq 4$. If $t \leq 3$, then $\dim(Q) \leq 2$ and Lemma 9 yields $|\text{vert}(P)| \leq 2^{n+1} - 2$. For $k \leq 3$ one has $t \leq 3$, so that the latter considerations already imply $g(\mathbb{Z}^n, k) \leq 2^{n+1} - 2$.

To get the desired upper bound for $k = 4$, the further possible value $t = 4$ needs to be addressed. For $t = 4$, the polytope Q is either a quadrilateral or a tetrahedron. If Q is a quadrilateral, Lemma 9 yields $|\text{vert}(P)| \leq 2^{n+1}$.

For the rest of the proof, let Q be a tetrahedron. Observe that $\mathbb{Z}^n \cap Q = X$. We use partitioning of \mathbb{Z}^n into 2^n residue classes modulo 2. Two points $x \in \mathbb{Z}^n$ and $y \in \mathbb{Z}^n$ are said to be in the same residue class modulo 2 if $x - y \in 2\mathbb{Z}^n$. Indexing the residue classes by $i \in [2^n]$, we denote by V_i the set of all vertices of P that fall into the i -th class.

With each V_i we associate the set $M_i := \{(v + w)/2 : v, w \in V_i, v \neq w\}$ consisting of all midpoints between pairs of distinct points in V_i . By the choice of V_i , one has $M_i \subseteq X$. In what follows, we bound the cardinalities of the sets V_i to get the desired bound on $|\text{vert}(P)|$.

First of all, every V_i contains at most three points, for otherwise M_i would have cardinality at least five, which is a contradiction to $|X| = k = 4$.

If the elements of V_i are congruent modulo 2 to some vertex of Q , we even get $|V_i| \leq 1$. In fact, assume that V_i contains two distinct points u, v and let w be the vertex of Q belonging to the same residue class as u and v . Then, the convex hull of u, v and w is either a line segment containing at least three integer points in its relative interior, or it is a triangle whose vertex u and the three midpoints of the edges are integer points. Thus, we find either three non-vertex integer points on a line or four non-vertex integer points in a two-dimensional affine space. Both cases contradict the properties of Q .

Finally, observe that there are at most four sets V_i with exactly three points. In fact, if V_i consists of exactly three points, then $\text{conv}(V_i)$ is a triangle with integer vertices. This implies that $\text{conv}(M_i)$ is a triangle with integer vertices, too. Since $M_i \subseteq X$, we conclude that $\text{conv}(M_i)$ is a facet of Q . Taking into account that a triangle is uniquely determined by its edge midpoints and that Q has four facets, we get that there are at most four sets V_i with $|V_i| = 3$.

Summarizing, we obtain that four of the sets V_i have cardinality at most 1, at most four of the sets V_i have cardinality 3 and all remaining sets have cardinality 2. This yields

$$|\text{vert}(P)| = \sum_{i=1}^{2^n} |V_i| \leq 4 \cdot 1 + 4 \cdot 3 + (2^n - 8) \cdot 2 = 2^{n+1}$$

and concludes the proof. \square

APPENDIX A. MAXIMAL CONVEX SETS WITH k POINTS OF S IN THE INTERIOR

This section presents another interpretation of $c(S, k)$ in the case that $S \subseteq \mathbb{R}^n$ is discrete and $k \in \mathbb{N}_0$ is arbitrary. We introduce the family $\mathcal{M}(S, k)$ of all inclusion-maximal n -dimensional convex sets with precisely k points of S in the interior. More formally, $M \in \mathcal{M}(S, k)$ if and only if M is an n -dimensional convex set with $|\text{int}(M) \cap S| = k$ such that for every convex set M' satisfying $M \subseteq M'$ and $\text{int}(M') \cap S = \text{int}(M) \cap S$ one necessarily has $M = M'$.

In optimization, sets $M \in \mathcal{M}(S, 0)$ are called *maximal S -free*; see [4], [10, Sect. 2] and [29]. Various families of maximal \mathbb{Z}^n -free sets have been extensively used for the generation of so-called *cutting planes*; see the survey [14]. Cutting planes are employed for gradually approximating a given mixed-integer problem with linear optimization problems; they belong to the standard tools for solving general mixed-integer problems. Note that, apart from $S = \mathbb{Z}^n$, also other choices of S such as $S = \mathbb{N}_0^n$ are of interest for mixed-integer optimization; see [19]. The possibility of using more general sets $M \in \mathcal{M}(S, k)$, where $k \in \mathbb{N}_0$ is arbitrary, similarly to maximal S -free sets is supported by the following simple observation. If $S = \mathbb{Z}^n$ and it is known that a particular point $z \in \mathbb{Z}^n$ does not correspond to a solution of the underlying mixed-integer problem, one can use sets $M \in \mathcal{M}(S, k)$ with $k = 1$ and $\text{int}(M) \cap \mathbb{Z}^n = \{z\}$ for the generation of cutting planes. Sets $M \in \mathcal{M}(S, k)$ for larger parameters k can be used analogously.

From the computational point of view, the complexity of generating cutting planes from $M \in \mathcal{M}(S, k)$ depends on the number $|\text{fct}(M)|$ of facets of M (where we interpret $|\text{fct}(M)|$ as ∞ if M is not a polyhedron). Thus, describing the maximum of $|\text{fct}(M)|$ for $M \in \mathcal{M}(S, k)$ is of interest. Theorem 15 below shows that this maximum is exactly $c(S, k)$. In the proof of this result we use the following proposition:

Proposition 14. *Let $S \subseteq \mathbb{R}^n$ be discrete, $k \in \mathbb{N}_0$ and let C be an n -dimensional convex set with $|\text{int}(C) \cap S| = k$. Then there exists an $M \in \mathcal{M}(S, k)$ with $C \subseteq M$.*

Proof. The assertion can be derived by a direct application of Zorn's lemma. For $k = 0$ this was observed in [10, Sect. 2] and [29, Sect. 3]. \square

The following is a somewhat stronger version of Theorem 7 from the introduction.

Theorem 15. *Let $S \subseteq \mathbb{R}^n$ be discrete and let $k \in \mathbb{N}_0$. Then*

$$c(S, k) = \max \{ |\text{fct}(M)| : M \in \mathcal{M}(S, k) \},$$

where we interpret $|\text{fct}(M)|$ as ∞ if M is not a polyhedron.

Proof. Let $m := \max \{ |\text{fct}(M)| : M \in \mathcal{M}(S, k) \}$. We need to verify $c(S, k) = m$. If $c(S, k) = -\infty$, there are no convex sets that contain exactly k points of S . This implies $\mathcal{M}(S, k) = \emptyset$ so that both $c(S, k)$ and m are equal to $-\infty$. Let now $c(S, k) > -\infty$.

We verify $c(S, k) \geq m$. This inequality is trivial if $c(S, k) = \infty$ or $m = -\infty$. So, assume that $c(S, k)$ is finite and $m > -\infty$. Consider an arbitrary $M \in \mathcal{M}(S, k)$. We show $|\text{fct}(M)| \leq c(S, k)$ using a straightforward adaption of the argument in [4, Lem. 4.1] from the case $k = 0$ to the case of an arbitrary $k \in \mathbb{N}_0$. Let $X := \text{int}(M) \cap S$. There exist n -dimensional polytopes P_t with $t \in \mathbb{N}$ such that $P_t \subseteq P_{t+1}$ and $\text{int}(P_t) \cap S = X$ for every $t \in \mathbb{N}$ and $\bigcup_{t=1}^{\infty} P_t = \text{int}(M)$. Since P_t is the intersection of finitely many closed halfspaces, by Lemma 8 there exists a polyhedron Q_t with at most $c(S, k)$ facets satisfying $P_t \subseteq Q_t$ and $\text{int}(Q_t) \cap S = X$. For $t \rightarrow \infty$, the compactness argument from [4, Proof of Lem. 4.1] applies without any changes: The polyhedra Q_t , all having at most $c(S, k)$ facets, generate a

polyhedron Q with at most $c(S, k)$ facets that satisfies $\text{int}(Q) \cap S = X$ and $M \subseteq Q$. Since M is maximal, one must have $M = Q$. That is, M is a polyhedron.

It remains to verify $c(S, k) \leq m$. This is trivial if $m = \infty$. So, we assume that $m < \infty$ which means in particular that every set in $\mathcal{M}(S, k)$ is a polyhedron. In view of (6), it suffices to show that $t := |S \cap P| - k \leq m$ for every P as in (6). Clearly, $0 \leq t \leq |\text{vert}(P)|$. We fix arbitrary vertices $v_1, \dots, v_t \in S$ of P . Since S is discrete, we can enclose P into an n -dimensional polytope Q such that $P \cap S = Q \cap S$, $\text{bd}(Q) \cap S = \{v_1, \dots, v_t\}$ and v_1, \dots, v_t lie in the relative interior of pairwise distinct facets F_1, \dots, F_t of Q . By construction, $\text{int}(Q) \cap S = |S \cap P| - t = k$. Proposition 14 yields the existence of $M \in \mathcal{M}(S, k)$ with $Q \subseteq M$. Since $v_1, \dots, v_t \in S$ lie in the relative interior of pairwise distinct facets F_1, \dots, F_t of Q , these facets are subsets of pairwise distinct facets of M . This shows that M has at least t facets and concludes the proof of $c(S, k) \leq m$. \square

Since the choice $S = \mathbb{Z}^n$ is of particular interest, we conclude the section by describing the geometry of the sets $M \in \mathcal{M}(\mathbb{Z}^n, k)$ more precisely. In view of Theorem 15 and the fact that $c(\mathbb{Z}^n, k) < \infty$, each $M \in \mathcal{M}(\mathbb{Z}^n, k)$ is a polyhedron.

A geometric characterization of the polyhedra in $\mathcal{M}(\mathbb{Z}^n, 0)$ was presented by Lovász in [28]; see also [5, 10] for proofs of this characterization. In particular, it is known that the unbounded polyhedra in $\mathcal{M}(\mathbb{Z}^n, 0)$ can be described through the bounded polyhedra in $\mathcal{M}(\mathbb{Z}^i, 0)$ with $1 \leq i < n$. Furthermore, the bounded polyhedra in $\mathcal{M}(\mathbb{Z}^n, 0)$ are precisely those n -dimensional polytopes that have no interior integer point and at least one integer point in the relative interior of each facet. It turns out that for $k \geq 1$, all elements of $\mathcal{M}(\mathbb{Z}^n, k)$ are polytopes. We show the above description of polytopes in $\mathcal{M}(\mathbb{Z}^n, 0)$ can be carried over to $\mathcal{M}(\mathbb{Z}^n, k)$ without any changes.

Theorem 16. *Let $k \in \mathbb{N}$. Then $\mathcal{M}(\mathbb{Z}^n, k)$ is the set of all n -dimensional polytopes P with precisely k interior integer points such that the relative interior of every facet of P contains a point of \mathbb{Z}^n .*

Proof. In the proof we adapt the arguments from [5]. Let $M \in \mathcal{M}(\mathbb{Z}^n, k)$. Without loss of generality assume that $0 \in \text{int}(M)$.

We first verify the boundedness of M by adapting an argument from [5, Lem. 4]. Assume that M is not bounded. Then there exists a nonzero vector u such that $\alpha u \in \text{int}(M)$ for every $\alpha \geq 0$. We consider a ball B with center at the origin and such that $B \subseteq \text{int}(M)$. If $\lambda \geq 0$ is large enough, the 0-symmetric convex body $K := \frac{1}{k}(\lambda[-u, u] + B)$ has volume larger than 2^n . It follows, by Minkowski's first theorem (cf. [21, Sect. 22]), that $\text{int}(K)$ contains a nonzero integer vector z . Hence $kz \in \text{int}(\lambda[-u, u] + B)$. After possibly replacing z by $-z$, we can assume $kz \in \text{int}(\lambda[0, u] + B) \subseteq \text{int}(M)$. Thus, $0z, \dots, kz$ are $k+1$ integer points in $\text{int}(M)$, which is a contradiction to the choice of M .

Once the boundedness of M is established, the rest of the assertion follows by generalizing the argument from [5, Proof of Thm. 1] from the case $k = 0$ to the case of an arbitrary k in a straightforward manner. \square

APPENDIX B. AN EXPLICIT CONSTANT FOR ANDREWS' THEOREM

Let $n \in \mathbb{N}$ with $n \geq 2$. Following Andrews' proof and notation, it is shown in [3] that for every n -dimensional polytope $P \in \mathcal{P}(\mathbb{Z}^n)$ one has

$$\text{vol}(P) \geq \kappa(n) |\text{vert}(P)|^{\frac{n+1}{n-1}},$$

where

$$\kappa(n) = \frac{1}{2} \cdot 3^{-n} \gamma(n) (\kappa'(n))^{\frac{n}{n-1}}, \quad (\text{p. 278 in Thm.})$$

$$\gamma(n) = n^{-n} \cdot c_1(n), \quad (\text{p. 275 in Lem. 7})$$

$$\kappa'(n) = (n!)^{-\frac{n-1}{n}} \cdot (\xi(n))^{-\frac{n-1}{n}} \cdot \left(\frac{2\pi^{n/2}}{\Gamma(n/2)} \right)^{-\frac{1}{n}}, \quad (\text{p. 278 in Thm.})$$

$$c_1(n) = \frac{1}{n!} \sqrt{n+1} \left(\frac{(n-2)!}{\sqrt{n}} \right)^{\frac{n}{n-1}}, \quad (\text{p. 275 in Lem. 7})$$

$$\xi(n) = (n-1)^{2n} \left(\frac{n!}{n^{2n}} \right)^{\frac{1}{n-1}} ((n-1)!)^{\frac{1}{n-1}}, \quad (\text{p. 274 in Lem. 6})$$

and $\Gamma(\cdot)$ is the Gamma-function; the information in the parentheses gives the exact place where the reader can find the respective constant in [3]. Thus, the constant $\alpha(n)$ in Theorem 10 can be chosen as any number at least

$$(13) \quad (\kappa(n))^{-\frac{n-1}{n+1}} = \frac{2^{\frac{n-1}{n+1}} (3n)^{\frac{n(n-1)}{n+1}}}{(c_1(n))^{\frac{n-1}{n+1}} (\kappa'(n))^{\frac{n}{n+1}}}.$$

We give an upper bound on this value by using the following simple estimations. Since

$$\xi(n) \leq n^{2n} \cdot \left(\frac{n^n}{n^{2n}} \right)^{\frac{1}{n}} \cdot ((n-1)^{n-1})^{\frac{1}{n-1}} = n^{2n} \cdot \frac{1}{n} \cdot (n-1) \leq n^{2n},$$

we obtain

$$(14) \quad \kappa'(n) = \frac{1}{(n! \cdot \xi(n))^{\frac{n-1}{n}}} \cdot \frac{(\Gamma(n/2))^{\frac{1}{n}}}{\sqrt{2}\sqrt{\pi}} \geq \frac{\frac{1}{2}}{n! \cdot \xi(n) \cdot 4} \geq \frac{1}{8n^{3n}}.$$

Furthermore, we clearly have

$$(15) \quad c_1(n) \geq \frac{1}{n!} \sqrt{n+1} \frac{(n-2)!}{\sqrt{n}} \geq \frac{1}{n^2}.$$

Thus, plugging (14) and (15) into (13), we established

$$(\kappa(n))^{-\frac{n-1}{n+1}} \leq \frac{2^{\frac{n-1}{n+1}} (3n)^{\frac{n(n-1)}{n+1}}}{\left(\frac{1}{n^2}\right)^{\frac{n-1}{n+1}} \cdot \left(\frac{1}{8n^{3n}}\right)^{\frac{n}{n+1}}} \leq 2 \cdot (3n)^n \cdot n^2 \cdot 8n^{3n} \leq (3n)^{4n}.$$

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REFERENCES

- [1] Iskander Aliev, Robert Bassett, Jesús A. De Loera, and Quentin Louveaux. A Quantitative Doignon-Bell-Scarf Theorem. arXiv:1405.2480 (to appear in *Combinatorica*), 2014.
- [2] Nina Amenta, Jesús A. De Loera, and Pablo Soberón. Helly's theorem: New variations and applications. arXiv:1508.07606, 2015.

- [3] George E. Andrews. A lower bound for the volume of strictly convex bodies with many boundary lattice points. *Trans. Amer. Math. Soc.*, 106:270–279, 1963.
- [4] Gennadiy Averkov. On maximal S -free sets and the Helly number for the family of S -convex sets. *SIAM J. Discrete Math.*, 27(3):1610–1624, 2013.
- [5] Gennadiy Averkov. A proof of Lovász’s theorem on maximal lattice-free sets. *Beitr. Algebra Geom.*, 54(1):105–109, 2013.
- [6] Gennadiy Averkov and Robert Weismantel. Transversal numbers over subsets of linear spaces. *Adv. Geom.*, 12(1):19–28, 2012.
- [7] Imre Bárány. Extremal problems for convex lattice polytopes: a survey. *Contemp. Math.*, 453:87–103, 2008.
- [8] Alexander Barvinok. *A Course in Convexity*, volume 54 of *Graduate Studies in Mathematics*. American Mathematical Society Providence, 2002.
- [9] Amitabh Basu and Timm Oertel. Centerpoints: A link between optimization and convex geometry. arXiv:1511.08609, 2015.
- [10] Amitabh Basu, Michele Conforti, Gérard Cornuéjols, and Giacomo Zambelli. Maximal lattice-free convex sets in linear subspaces. *Math. Oper. Res.*, 35(3):704–720, 2010.
- [11] David E. Bell. A theorem concerning the integer lattice. *Studies in Appl. Math.*, 56(2):187–188, 1977.
- [12] Wouter Castryck. Moving out the edges of a lattice polygon. *Discrete Comput. Geom.*, 47(3):496–518, 2012.
- [13] Stephen R. Chestnut, Robert Hildebrand, and Rico Zenklusen. Sublinear Bounds for a Quantitative Doignon-Bell-Scarf Theorem. arXiv:1512.07126, 2015.
- [14] Michele Conforti, Gérard Cornuéjols, and Giacomo Zambelli. A geometric perspective on lifting. *Oper. Res.*, 59(3):569–577, 2011.
- [15] Jesús A. De Loera, Reuben N. La Haye, Deborah Oliveros, and Edgardo Roldán-Pensado. Helly numbers of Algebraic Subsets of \mathbb{R}^d . arXiv:1508.02380, 2015.
- [16] Jesús A. De Loera, Reuben N. La Haye, Deborah Oliveros, and Edgardo Roldán-Pensado. Beyond Chance-Constrained Convex Mixed-Integer Optimization: A Generalized Calafiore-Campi Algorithm and the notion of S -optimization. arXiv:1504.00076, 2015.
- [17] Jesús A. De Loera, Reuben N. La Haye, David Rolnick, and Pablo Soberón. Quantitative Tverberg, Helly, & Carathéodory theorems. arXiv:1503.06116, 2015.
- [18] Jean-Paul Doignon. Convexity in crystallographical lattices. *J. Geom.*, 3:71–85, 1973.
- [19] Ricardo Fukasawa and Oktay Günlük. Strengthening lattice-free cuts using non-negativity. *Discrete Optim.*, 8(2):229–245, 2011.
- [20] Bernardo González Merino and Matthias Henze. A generalization of the discrete version of Minkowski’s fundamental theorem. arXiv:1412.3315 (to appear in *Mathematika*), 2014.
- [21] Peter M. Gruber. *Convex and Discrete Geometry*, volume 336 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 2007.
- [22] Peter M. Gruber and Cornelis G. Lekkerkerker. *Geometry of Numbers*, volume 37 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, second edition, 1987.
- [23] Branko Grünbaum. The dimension of intersections of convex sets. *Pacific J. Math.*, 12:197–202, 1962.
- [24] Branko Grünbaum. Measures of symmetry for convex sets. In *Proc. Sympos. Pure Math., Vol. VII*, pages 233–270. Amer. Math. Soc., Providence, R.I., 1963.

- [25] Eduard Helly. Über Mengen konvexer Körper mit gemeinschaftlichen Punkten. *Jahresber. Dtsch. Math. Ver.*, 32:175–176, 1923.
- [26] Alan J. Hoffman. Binding constraints and Helly numbers. In *Second International Conference on Combinatorial Mathematics (New York, 1978)*, volume 319 of *Ann. New York Acad. Sci.*, pages 284–288. New York Acad. Sci., New York, 1979.
- [27] Jeffrey C. Lagarias and Günter M. Ziegler. Bounds for lattice polytopes containing a fixed number of interior points in a sublattice. *Canad. J. Math.*, 43(5):1022–1035, 1991.
- [28] László Lovász. Geometry of numbers and integer programming. In *Mathematical programming (Tokyo, 1988)*, volume 6 of *Math. Appl. (Japanese Ser.)*, pages 177–201. SCIPRESS, Tokyo, 1989.
- [29] Diego A. Morán Ramirez and Santanu S. Dey. On maximal S -free convex sets. *SIAM J. Discrete Math.*, 25(1):379–393, 2011.
- [30] Herbert E. Scarf. An observation on the structure of production sets with indivisibilities. *Proc. Nat. Acad. Sci. U.S.A.*, 74(9):3637–3641, 1977.
- [31] Rolf Schneider. *Convex Bodies: The Brunn-Minkowski Theory*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 2nd edition, 2014.